EMBEDDED SPHERES IN $S^2 \times S^1 \# \dots \# S^2 \times S^1$

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ABSTRACT. We give an algorithm to decide which elements of $\pi_2(\#_k S^2 \times S^1)$ can be represented by embedded spheres. Such spheres correspond to splittings of the free group F_k on k generators. Equivalently our algorithm decides whether, for a handlebody N, an element in $\pi_2(N, \partial N)$ can be represented by an embedded disc. We also give an algorithm to decide when classes in $\pi_2(\#_k S^2 \times S^1)$ can be represented by disjoint embedded spheres.

We introduce the splitting complex of a free group which is analogous to the complex of curves of a surface. We show that the splitting complex of $\mathbb{F}_k$ embeds in the complex of curves of a surface of genus k as a quasi-convex subset.

1. Introduction

We study here embedded spheres in a 3-manifold of the form $M = \#_k(S^2 \times S^1)$, i.e., the connected sum of k copies of $S^2 \times S^1$. Group theoretically such spheres correspond to splittings of the free group $\mathbb{F}_k$ on k generators [10]. Understanding these is likely to be useful in studying $Out(\mathbb{F}_k)$, which is the mapping class group of the M, and more generally the mapping class group of reducible 3-manifolds.

Splittings of free groups also correspond to properly embedded discs in handlebodies [3][5]. Hence all our results can be rephrased in terms of properly embedded discs in handlebodies. Because of the relation to Heegaard splittings, our results are likely to be useful in this formulation. However, for simplicity of notation, we shall consider spheres in M.

The first question we consider is whether a class in $\pi_2(M)$ can be represented by an embedded sphere in M. Let $\tilde{M}$ be the universal cover of M. Observe that $\pi_2(M) = \pi_2(\tilde{M}) = H_2(\tilde{M})$ by Hurewicz theorem. We shall implicitly use this identification throughout.

We first consider when $A \in H_2(\tilde{M}) = \pi_2(M)$ can be represented by an embedded sphere in $\tilde{M}$. We shall make use of intersection numbers (and Poincaré duality) for non-compact manifolds. Represent A by a (not necessarily connected) surface in $\tilde{M}$ (also denoted A). Given a proper map $c : \mathbb{R} \to \tilde{M}$ which is transversal to A, we consider the algebraic intersection number $c \cdot A$. This depends only on the homology class of A and the proper homology class of c. The following gives a criterion for A to be represented by an embedded sphere.

Theorem 1.1. *The class $A \in H_2(\tilde{M})$ can be represented by an embedded sphere if and only if for each proper map $c : \mathbb{R} \to \tilde{M}$, $c \cdot A \in \{0, 1, -1\}$.*

For an embedded sphere $S \in M$ with lift $\tilde{S} \in M$, all the translates of $\tilde{S}$ are disjoint from $\tilde{S}$. In particular, if $A = [\tilde{S}]$ is the class represented by $\tilde{S}$, then A and

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gA can be represented by disjoint spheres for each deck transformation g. Thus, our next step is to give a criterion for when two classes A and B in $H_2(\tilde{M})$ can be represented by disjoint spheres.

Theorem 1.2. *Let A and B be classes in $H_2(\tilde{M})$ that can be represented by embedded spheres. Then A and B can be represented by disjoint embedded spheres if and only if there do not exist proper maps $c, c' : \mathbb{R} \to \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$.*

The two above theorems let us determine when, for a class $A \in \pi_2(M) = H_2(\tilde{M})$, the homology classes A and gA can be represented by disjoint spheres for each $g \in \pi_1(M)$. However to get an embedded sphere in M, we need more. Namely, such a sphere S exists if and only if there is a sphere $\tilde{S}$ disjoint from all its translates $g\tilde{S}$.

Our next result shows that this is automatically satisfied.

Theorem 1.3. *Suppose $A \in \pi_2(M) = H_2(\tilde{M})$ is a class such that each for each $g \in \pi_1(M)$, A and gA can be represented by disjoint spheres in $\tilde{M}$. Then A can be represented by an embedded sphere $S \in M$.*

Thus, we have a criterion for deciding which class can be represented by an embedded sphere. However our criterion *a priori* involves checking conditions for infinitely many proper maps $c, c' : \mathbb{R} \to \tilde{M}$ and infinitely many group elements g. We shall show that it suffices to check only finitely many conditions. This gives the following result.

Theorem 1.4. *There is an algorithm that decides whether a class $A \in \pi_2(M)$ can be represented by an embedded sphere in M.*

Our methods extend to deciding when two classes A and B can be represented by disjoint spheres in M. This is based on an analogue of Theorem 1.3.

Theorem 1.5. *Suppose A and B are classes in $\pi_2(M)$ that can be represented by embedded spheres in M. Then A and B can be represented by disjoint spheres in M if and only if for each $g \in \pi_1(M)$, A and gB can be represented by disjoint spheres in $\tilde{M}$.*

Theorem 1.6. *There is an algorithm that decides whether classes $A, B \in \pi_2(M)$ can be represented by disjoint embedded spheres in M.*

In group theoretic terms, isotopy classes of embedded spheres in M correspond to conjugacy classes of splittings of the free group $\mathbb{F}_k$. Disjoint spheres in M correspond to splittings compatible up to conjugacy.

We define the *splitting complex* of $\mathbb{F}_k$ in a manner analogous to the complex of curves, which has proved very useful in the study of the mapping class group [2][4] as well as 3-manifold topology [7]. Namely, we consider a simplicial complex with vertices corresponding to conjugacy classes of splittings of $\mathbb{F}_k$. A finite set of vertices bounds a simplex if the corresponding splittings are compatible up to conjugacy. This gives a simplicial complex.

We shall see that this is a (quasi-convex) subcomplex of the complex of curves.

Theorem 1.7. *The splitting complex of $\mathbb{F}_k$ is isomorphic to a subcomplex of the complex of curves of a surface of genus k. Further this subcomplex is a quasi-convex subset of the complex of curves.*

The construction of the splitting complex can be made for an arbitrary group. Moreover, we can consider splittings over any class of subgroups, for example polycyclic groups. Indeed the complex of curves is the splitting complex of a surface group over $\mathbb{Z}$.

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2. Ends and spheres in $\tilde{M}$

We recall the notion of ends of a space. Let X be a topological space. For a compact set $K \subset X$, let $C(K)$ denote the set of components of $X - K$. For L compact with $K \subset L$, we have a natural map $C(L) \to C(K)$. Thus, as compact subsets of X define a directed system under inclusion, we can define the set of ends $E(X)$ as the inverse limit of the sets $C(K)$.

It is easy to see that a proper map $f : X \to Y$ induces a map $E(X) \to E(Y)$ and that this is functorial. In particular, the real line $\mathbb{R}$ has two ends which can be regarded as $-\infty$ and ∞. Hence a proper map $c : \mathbb{R} \to X$ gives a pair of ends c_- and c_+ of X.

Now consider proper maps $c : \mathbb{R} \to \tilde{M}$. As $\tilde{M}$ is a union of simply connected compact sets, the following lemma is straightforward.

Lemma 2.1. *There is a one-one correspondence between proper homotopy classes of maps $c : \mathbb{R} \to \tilde{M}$ and pairs $(c_-, c_+) \in E(\tilde{M}) \times E(\tilde{M})$*

We shall refer to a curve c as above as a proper path from c_- to c_+ or as a proper path joining c_- and c_+. We denote such a path c by (c_-, c_+). This is well defined up to proper homotopy. In particular, for a homology class $A \in H_2(\tilde{M})$, the intersection number $(c_-, c_+) \cdot A$ is well defined and can be computed using any proper path joining c_- and c_+. We shall use this implicitly throughout.

We now characterise which homology classes in $\tilde{M}$ can be represented by embedded spheres.

Proof of Theorem 1.1. Suppose A can be represented by an embedded sphere S. Then the complement of S consists of two components with closures X_1 and X_2. As S is compact, the space of ends of $\tilde{M}$ is also partitioned into sets $E_i = E(X_i)$. For a pair of ends (c_-, c_+), if both c_- and c_+ are contained in the same E_i, we have a corresponding proper path c disjoint from S. Otherwise we can choose c intersecting S in one point. In either case, $c \cdot A$ is 0, 1 or -1. Computing intersection numbers $(c_-, c_+) \cdot A$ using these paths, it follows that $c \cdot A$ is always 0, 1 or -1.

Conversely, assume that for each $c = (c_-, c_+)$, $c \cdot A$ is one of 0, 1 or -1. Let A be represented by a (not necessarily connected) smooth, closed surface, which we also denote A. Let $K \supset A$ be a compact, 3-dimensional, connected manifold contained in $\tilde{M}$ such that the closure W_i of each complementary component of K is non-compact. As $\tilde{M}$ is simply-connected and K is connected, $N_i = \partial W_i$ is connected for each W_i. Note that there are finitely many sets W_i and $E(\tilde{M})$ is partitioned into the sets $E(W_i)$.

We define a relation on the space of ends $E(\tilde{M})$ as follows. For a pair of ends e_0 and e_1, let c be a proper path joining e_0 to e_1. We define $e_0 \sim e_1$ if $c \cdot A = 0$. We shall show that the relation $\sim$ is an equivalence relation. When $A \neq 0$ we show that there are exactly two equivalence classes.

We first need a lemma.

Lemma 2.2. *For ends e, f and g of $\tilde{M}$.*

- $(e,f)\cdot A = -(f,e)\cdot A$
- $(e,g)\cdot A = (e,f)\cdot A + (f,g)\cdot A$

Proof. The first part is immediate from the definitions. Suppose now e, f and g are ends and let c and c' be proper paths from e to f and from f to g respectively. Let k be such that $f \in E(W_k)$. Then there exist $T \in R$ such that $c([T,\infty)) \subset W_k$ and $c'((-\infty,-T]) \subset W_k$. Let γ be a path in W_k joining $c(T)$ and $c'(-T)$. Consider the path $c'' = c|_{(-\infty,T]} * \gamma * c'|_{[-T,\infty)} : \mathbb{R} \to \tilde{M}$. This is a proper path from e to g and its intersection points with A are the union of those of c with A and c' with A, with the signs associated to the points of $c'' \cap A$ agreeing with the signs for $c \cap A$ and $c' \cap A$. Computing $(e,g)\cdot A$ using c'', we see $(e,g)\cdot A = (e,f)\cdot A + (f,g)\cdot A$ as claimed. $\square$

By the above, $\sim$ is an equivalence relation. We next show that there at most most two equivalence classes. This follows from the next lemma.

Lemma 2.3. *Suppose $e \not\sim f$ and $e \not\sim g$. Then $f \sim g$ and $(e,f)\cdot A = (e,g)\cdot A$.*

Proof. By Lemma 2.2, we have

$$(f,g)\cdot A = (e,g)\cdot A - (e,f)\cdot A$$

By hypothesis, each of $(e,g)\cdot A$ and $(e,f)\cdot A$ is ± 1 and their difference $(f,g)\cdot A$ is 0, 1 or -1. It follows that $(e,f)\cdot A = (e,g)\cdot A$ and $(f,g)\cdot A = 0$, i.e., $f \sim g$. $\square$

Now as $A \neq 0$ in homology, by Poincaré duality there are ends e and f such that $(e,f)\cdot A \neq 0$, i.e., $e \not\sim f$. Thus there are exactly two equivalence classes of ends which we denote E_1 and E_2.

Next, observe that given two points in $E(W_i)$, for some i, there is a path joining these in the complement of K, hence of A. It follows that these are equivalent. Hence each $E(W_i)$ is contained in E_1 or E_2. We now construct a proper function $f : \tilde{M} \to \mathbb{R}$. Namely, for each i, if $E(W_i) \subset E_1$ (respectively $E(W_i) \subset E_2$), we construct a proper function $f : W_i \to [-1,-\infty)$ (respectively $f : W_i \to [1,\infty)$). We extend this across K to get a proper function $f : \tilde{M} \to \mathbb{R}$.

As $\tilde{M}$ is simply-connected, using standard techniques due to Whitehead and Stallings [9][10], after a proper homotopy of f we can assume that $S = f^{-1}(0)$ is a sphere. This separates $\tilde{M}$ into subsets X_1 and X_2. By construction, $E(X_i) = E_i$. Hence by Poincaré duality, after possibly changing the orientation of S, $A = [S]$ as claimed. $\square$

Remark 2.4. By construction $S \subset K$.

3. Disjoint spheres in $\tilde{M}$

Suppose now that A and B are classes in $H_2(\tilde{M}) = \pi_2(M)$ which can be represented by embedded spheres S and T. We consider the when S and T can be chosen to be disjoint. Denote the closures of the components of the complement of S (respectively T) by X_1 and X_2 (respectively Y_1 and Y_2) so that $(e,f)\cdot A = 1$ if and only if $e \in X_1$ and $f \in X_2$ and $(e,f)\cdot B = 1$ if and only if $e \in Y_1$ and $f \in Y_2$. Recall that $(f,e)\cdot A = -(e,f)\cdot A$ and $(f,e)\cdot B = -(e,f)\cdot B$.

Suppose S and T are disjoint. We first consider the case $T \subset X_2$. Then X_1 is contained in one of Y_1 and Y_2. If $X_1 \subset Y_1$, then for $c' = (e, f)$, if $c'\dot{A} = 1$ then $e \in X_1 \subset Y_1$ hence $(f, e) \cdot B \neq 1$, i.e., $c' \cdot B \neq -1$. Thus, there does not exist c' with $c' \cdot A = 1 = -c' \cdot B$.

By considering other cases similarly, we see that there do not exist proper maps $c, c' : \mathbb{R} \to \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$.

Conversely, suppose there do not exist proper maps $c, c' : \mathbb{R} \to \tilde{M}$ with $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$. We define three equivalence relations $\sim_A$, $\sim_B$ and $\sim$ on $E(\tilde{M})$. Namely, $e \sim_A f$ (respectively $e \sim_B f$) if $(e, f) \cdot A = 0$ (respectively $(e, f) \cdot B = 0$) and $e \sim f$ if $e \sim_A f$ and $e \sim_B f$. We shall see that $\sim$ partitions $E(\tilde{M})$ into at most three equivalence classes.

Let $e \in E(\tilde{M})$ be an end. By Lemma 2.3, for ends f, $(e, f) \cdot A$ has only two possible values, 0 and one of 1 and -1. By replacing A by $-A$, we assume $(e, f) \cdot A$ is always 0 or 1. Similarly, we assume that $(e, f) \cdot B$ is always 0 or 1. Thus, for ends f, the pair$((e, f) \cdot A, (e, f) \cdot B)$ has four possible values. By Lemma 2.2, if $((e, f) \cdot A, (e, f) \cdot B) = ((e, g) \cdot A, (e, g) \cdot B)$, then $f \sim g$. Hence there are at most four equivalences classes under the relation $\sim$.

We need to show that at least one of these classes is empty. If not, we can find f, g and h with $(e, f) \cdot A = 1$, $(e, f) \cdot B = 0$, $(e, g) \cdot A = 0$, $(e, g) \cdot B = 1$, $(e, h) \cdot B = 1$ and $(e, h) \cdot B = 1$. Taking $c = (e, h)$ and $c' = (g, f)$, by Lemma 2.2 we see that $c \cdot A = 1 = c \cdot B$ and $c' \cdot A = 1 = -c' \cdot B$, a contradiction.

Remark 3.1. As the four equivalence classes under $\sim$ are the four intersections $E(X_i) \cap E(Y_j)$, we see that one of these sets must be empty, i.e. one of the sets $X_i \cap Y_j$ is compact. This is important in the sequel.

If A and B are not independent, then either $A = B$ or $A = -B$ as both A and B are represented by embedded spheres and are hence primitive. In this case they can be represented by disjoint embedded spheres. Hence we may assume that they are independent. By Poincaré duality, it follows that there must be three equivalence classes. Let e, f and g represent the equivalence classes. By changing signs and permuting if necessary, we can assume that $(e, f) \cdot A = 1$, $(e, f) \cdot B = 0$, $(e, g) \cdot B = 0$ and $(e, g) \cdot B = 1$.

We now proceed as in the previous section. Choose surfaces representing A and B and a compact submanifold K containing these as in the previous section. Let T (a tripod) denote the union of three half lines R_e, R_f and R_g, each homeomorphic to $[0, \infty)$, with the points 0 in all of them identified. We construct a proper map $f : \tilde{M} \to T$ by mapping the components of W_i equivalent to e properly onto T_e and analogously for the other components and extending this over K. Let $1_f \in R_f$ and $1_g \in R_g$ denote points corresponding to 1. Then using the techniques of Whitehead and Stallings, after a proper homotopy of f, $S = f^{-1}(1_e)$ and $T = f^{-1}(1_h)$ are disjoint spheres representing A and B.

4. Intersection numbers and embedded Spheres

Suppose now that the class $A \in \pi_2(M) = H_2(\tilde{M})$ can be represented by an embedded sphere S in $\tilde{M}$. Further assume that for all $g \in \pi_1(M)$, A and gA can be represented by disjoint embedded spheres. We show that the class A is represented by a splitting of the free group $G = \pi_1(M)$ and hence an embedded sphere.

This follows from the work of Scott and Swarup [8] using Remark 3.1. In our case we only consider splittings over the trivial group which simplifies our considerations.

Let X_1 and X_2 be the closures of the complementary components of S. The Cayley graph of G embeds in $\tilde{M}$ and the vertices can be identified with elements of G. Let $E_i = X_i \cap G$. Then E_1 and E_2 form (almost) complementary almost-invariant sets. The self-intersection number of the set E_1 is the number of $g \in G$ such that all the four sets $E_i \cap gE_j$, $i, j \in \{1, 2\}$, are infinite. But by Remark 3.1, for each $g \in G$ at least one of the intersections $X_i \cap gX_j$ is compact which implies that the corresponding intersection $E_i \cap gE_j$ is finite (as G is a discrete subset of $\tilde{M}$). Thus the self-intersection number of E_1 is zero.

By a result of Scott and Swarup [8], it follows that there is a splitting of the group G corresponding to A. Hence, by the Knesser conjecture, there is an embedded sphere representing the class A. This completes the proof of Theorem 1.3

The proof of Theorem 1.5 is very similar. We use the result of Scott and Swarup [8] that two splittings are compatible if the intersection number between the corresponding almost invariant sets vanishes.

5. The Algorithms

We now have necessary and sufficient conditions for deciding whether a class $A \in \pi_2(M)$ can be represented by an embedded sphere in M. However there are *a priori* infinitely many conditions. To make this into an algorithm, we reduce these to finitely many conditions.

Firstly, let $\Gamma \subset M$ be a wedge of circles dual to the spheres in $M = \#_k S^2 \times S^1$. Then the universal cover T of Γ is a tree which embeds in $\tilde{M}$. We observe that the complement of the spheres lifts to a set in $\tilde{M}$ with closure P a fundamental domain. P intersects T in a unique vertex and each vertex of T is contained in a unique translate of P.

Any proper path c is properly homotopic to an edge path in T. Further, $\pi_2(M) = H_2(\tilde{M})$ is generated by spheres S which intersect exactly one edge e of T and with $S \cap e$ is a single point with transversal intersection.

Thus, elements of $\pi_2(M)$ correspond to finite linear combinations of edges of T. Let A be such an element, and let $\tau \subset T$ be a finite subtree containing the support of A. Then for an edge-path c, $c \cdot A$ depends only on the finite edge path $\xi = c \cap \tau$ contained in τ with endpoints on $\partial\tau$. Further, as T is a tree without any terminal vertices, any finite edge path ξ in τ with endpoints on $\partial\tau$ is of the form $\xi = c \cap \tau$ for a proper path c. Hence A is represented by an embedded sphere in $\tilde{M}$ if and only if for every finite edge path ξ in τ with endpoints on $\partial\tau$, $\xi \cdot A$ is 0, 1 or -1.

Similarly, given two homology classes A and B in $H_2(\tilde{M})$, we have an algorithm to decide whether A and B can be represented by disjoint embedded spheres by taking τ containing the supports of both A and B.

Finally, if A is a homology class with τ a tree supporting A, we first verify whether A can be embedded in $\tilde{M}$. Next there are at most finitely many elements $g_1, \dots g_n$ in G such that $\tau \cap g_i\tau$ is non-empty. For each of these g_i we check whether A and g_iA can be represented by disjoint spheres. Assume henceforth that A has this property.

Let K be the union of the translates of P containing vertices of τ. Then K is as in the proof of Theorem 1.1. Thus A can be represented by an embedded sphere S in K. If $\tau \cap g\tau$ is empty, so is $K \cap gK$ and hence $S \cap gS$, i.e. A and gA can

be represented by disjoint embedded spheres. Thus we need to check only finitely many conditions for finitely many g_i, which can be done algorithmically.

Similar considerations, using Theorem 1.5 gives an algorithm to decide whether two classes in $\pi_2(M)$ (more generally finitely many classes in $\pi_2(M)$) can be represented by disjoint spheres.

6. The Splitting Complex

The *complex of curves* $CS(S)$ of a surface S has proved to be very useful in studying the mapping class group of a surface as well as in 3-manifold topology. We define analogously the *splitting complex* $SC(\mathbb{F}_k)$ of a free group $\mathbb{F}_k$.

Namely, let V be the set of splittings of $\mathbb{F}_k$ up to conjugacy, or equivalently the set of properly embedded discs in a handlebody H_k of genus k up to isotopy. To define a simplicial complex with vertices V, it suffices to specify when a finite subset of V, i.e. a finite collection of splittings, is the set of vertices of a simplex. We define the splitting complex by specifying that a collection of splittings bounds a simplex if it is compatible up to conjugacy. In topological terms vertices corresponding to a collection of embedded disjoint discs in H_k bound a simplex in $SC(\mathbb{F}_n)$ if they are isotopic to disjoint embedded discs.

Theorem 6.1. *Let S_k be the surface of genus k. Then $SC(\mathbb{F}_k)$ is isomorphic to a connected quasi-convex subcomplex of $CS(S_k)$.*

Proof. We interpret $SC(\mathbb{F}_k)$ in terms of discs in H_k. Associating to each disc its boundary gives an embedding of $SC(\mathbb{F}_k)$ in $CS(\mathbb{F}_k)$. By results of Masur and Minsky [6] it follows that the image is connected and quasi-convex. □

Many fruitful results regarding the complex of curves, in particular [1], have been obtained by studying the relation between distances in the complex of curves and intersection numbers. Thus one may hope that similar results regarding the splitting comlex (and hence $Out(\mathbb{F}_k)$) may be obtained using our methods. A particularly interseting question is hyperbolicity of the splitting complex.

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